4 Posets and lattices

4.1 Partial orders

Many important relations cover some idea of greater and smaller: the *partial order*s.

4.1 **Definition.** An (endo)relation $\sqsubseteq$ ("under") on a set P is called a *partial order* if it is reflexive, antisymmetric, and transitive. We recall that this means that, for all $x, y, z \in P$, we have:

- $x \sqsubseteq x$;
- $x \sqsubseteq y \ \wedge \ y \sqsubseteq x \ \Rightarrow \ x = y$;
- $x \sqsubseteq y \ \wedge \ y \sqsubseteq z \ \Rightarrow \ x \sqsubseteq z$.

The pair $(P, \sqsubseteq)$ is called a *partially ordered set* or, for short, a *poset*.

Two elements x and y in a poset $(P, \sqsubseteq)$ are called *comparable* if $x \sqsubseteq y$ or $y \sqsubseteq x$, otherwise they are called *incomparable*, that is, if $\neg(x \sqsubseteq y)$ and $\neg(y \sqsubseteq x)$.

A partial order is a *total order*, also called *linear order*, if every two elements are comparable.

□

4.2 **Lemma.** For every poset $(P, \sqsubseteq)$ and for every subset X of P, the pair $(X, \sqsubseteq)$ is a poset too.

□

4.3 **Example.**

- On every set, the identity relation I is a partial order. It is the *smallest* possible partial order relation on that set.
- On the real numbers $\mathbb{R}$ the relation $\leq$ is a total order: every two numbers $x, y \in \mathbb{R}$ satisfy $x \leq y$ or $y \leq x$. Restriction of $\leq$ to any subset of $\mathbb{R}$ – for example, restriction to $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$ – also yields a total order on that subset.
- The power set $\mathcal{P}(V)$ of a set V, that is, the set of all subsets of V, with relation $\subseteq$ (subset inclusion), is a poset. This P contains a smallest element, namely $\emptyset$, and a largest element, namely V itself.
- The relation $|$ ("divides") is a partial order on the positive naturals, defined by

$$x \mid y \ \Leftrightarrow \ (\exists z : z \in \mathbb{N} : x * z = y).$$

□

4.4 **Definition.** The *irreflexive part* of a partial order relation $\sqsubseteq$ is denoted by $\sqsubset$ ("strictly under") and is defined by, for all $x, y \in P$:

$$x \sqsubset y \;\Leftrightarrow\; x \sqsubseteq y \,\wedge\, x \neq y \;.$$

□

It is straightforward to prove that the relation $\sqsubset$ is irreflexive, antisymmetric and transitive. Further, it directly follows from the definition that for all $x, y \in P$: $x \sqsubseteq y \;\Leftrightarrow\; x \sqsubset y \,\vee\, x = y$.

4.5 Lemma. If $(P, \sqsubseteq)$ is a poset, then the corresponding directed graph, with vertex set P and arrows (x, y) whenever $x \sqsubset y$, is *acyclic*.
□

If we want to draw a picture of a finite poset, with the greater elements on top and the smaller elements below, and an arrow from x to y if $x \sqsubseteq y$ holds, we usually do not draw the whole graph. Edges from a node x to itself, representing $x \sqsubseteq x$ are not drawn, and an edge from x to y with $x \sqsubseteq y$ is only drawn if there is no z, distinct from both x and y, for which we have $x \sqsubseteq z$ and $z \sqsubseteq y$. The resulting directed graph is called the *Hasse diagram* for $(P, \sqsubseteq)$, named after the German mathematician Helmut Hasse (1898-1979). Hasse diagrams are drawn in such a way that two vertexes x and y with $x \sqsubseteq y$ are connected by an edge going upwards. In Theorem **??** we will see that this is always possible. Consequently, the edges in Hasse diagrams do not need an arrow point to indicate the direction. For example the Hasse diagram for the poset $(\mathcal{P}(\{a, b, c\}), \subseteq)$ is drawn as in Figure 23.

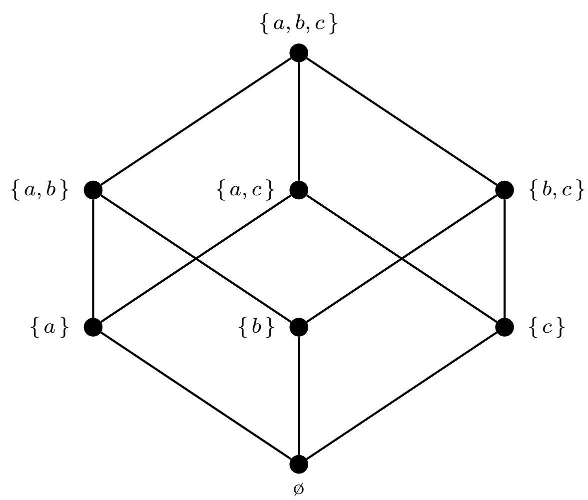

Figure 23: A Hasse diagram of $(\mathcal{P}(\{a, b, c\}), \subseteq)$

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There are various ways of constructing new posets out of old ones. We discuss some of them. In the sequel both $(P, \sqsubseteq)$ and $(Q, \sqsubseteq)$ are posets. Notice that we use the *same* symbol, $\sqsubseteq$, for the two *different* partial order relations on sets P and Q . If confusion may arise we distinguish the two relations by using $\sqsubseteq_P$ and $\sqsubseteq_Q$, respectively.

- As already stated in Lemma 4.2, for every subset X of P the pair $(X, \sqsubseteq)$ is a poset, with relation $\sqsubseteq$ restricted to X . Thus restricted $\sqsubseteq$ is called the *induced* order on X .

- Relation $\sqsupseteq$ ("above"), defined by, for all $x, y \in P$, $x \sqsupseteq y \Leftrightarrow y \sqsubseteq x$ is a partial order too, called the *dual order to* $\sqsubseteq$; hence, $(P, \sqsupseteq)$ also is a poset.

- Let V be a set. On the set $V \to P$ of functions from V to P we can define a partial order $\sqsubseteq_{VP}$, say, as follows, for all $f, g \in V \to P$:

  $$f \sqsubseteq_{VP} g \;\; \Leftrightarrow \;\; (\forall v : v \in V : f(v) \sqsubseteq_P g(v)) \quad .$$

  Then $(V \to P\,, \sqsubseteq_{VP})$ is a poset.

- On the Cartesian product $P \times Q$ we can define a partial order as follows. For $(p, q)\,, (x, y) \in P \times Q$ we define:

  $$(p, q) \sqsubseteq (x, y) \;\; \Leftrightarrow \;\; p \sqsubseteq_P x \;\wedge\; q \sqsubseteq_Q y \quad .$$

  Thus defined relation $\sqsubseteq$ is a partial order, called the *product order* on $P \times Q$.

- On the Cartesian product $P \times Q$ we also can define a partial order as follows. For $(p, q)\,, (x, y) \in P \times Q$ we define:

  $$(p, q) \sqsubseteq (x, y) \;\; \Leftrightarrow \;\; (p \neq x \wedge p \sqsubseteq_P x) \;\vee\; (p = x \wedge q \sqsubseteq_Q y) \quad .$$

  This relation $\sqsubseteq$ is a partial order too, called the *lexicographic order* on $P \times Q$.

The notions of product order and lexicographic order can be extended to (finite) products of more than two sets.

A poset's structure is determined by its partial order relation, as denoted by $\sqsubseteq$ in the previous section. Sometimes we wish to prove equalities in a poset, and then it can be convenient if we can reformulate an equality in terms of the partial order relation: that enables us to reason in terms of the partial order.

The following lemma provides such a connection between equality and the partial order.

4.6 Lemma. In every poset $(P, \sqsubseteq)$ we have, for all $x, y \in P$:

(a) $x = y \;\Leftrightarrow\; (\forall z : z \in P : x \sqsubseteq z \Leftrightarrow y \sqsubseteq z)$;

(b) $x = y \;\Leftrightarrow\; (\forall z : z \in P : z \sqsubseteq x \Leftrightarrow z \sqsubseteq y)$.

Proof. We prove (a) only; the proof of (b) follows, mutatis mutandis, the same pattern. We do so by mutual implication.

"$\Rightarrow$": Assuming $x=y$ we may substitute x for y and vice versa wherever we like; in particular, if we substitute x for y in our demonstrandum $(\forall z : z{\in}P : x{\sqsubseteq}z \Leftrightarrow y{\sqsubseteq}z)$, we obtain $(\forall z : z{\in}P : y{\sqsubseteq}z \Leftrightarrow y{\sqsubseteq}z)$, which is true because of the reflexivity of $\Leftrightarrow$.

"$\Leftarrow$": Assuming $(\forall z : z{\in}P : x{\sqsubseteq}z \Leftrightarrow y{\sqsubseteq}z)$, by the instantiation $z:=x$ we obtain $x\sqsubseteq x \Leftrightarrow y\sqsubseteq x$, which is equivalent to $y\sqsubseteq x$, because $\sqsubseteq$ is reflexive. Moreover, by the instantiation $z:=y$ we obtain $x\sqsubseteq y \Leftrightarrow y\sqsubseteq y$, which is equivalent to $x\sqsubseteq y$. By the antisymmetry of $\sqsubseteq$ we conclude $x=y$, as required.
□

4.2 Extreme elements

4.7 **Definition.** For any subset X of a poset $(P,\sqsubseteq)$ we define, for all $m{\in}P$:

(a) m is X's *maximum*, or *largest element* if:

$$m\in X \;\wedge\; (\forall x : x{\in}X : x{\sqsubseteq}m) \quad .$$

(b) m is X's *minimum*, or *least element* if:

$$m\in X \;\wedge\; (\forall x : x{\in}X : m{\sqsubseteq}x) \quad .$$

(c) m is *a maximal* element of X if:

$$m\in X \;\wedge\; (\forall x : x{\in}X : \neg(m{\sqsubset}x)) \quad .$$

(d) m is *a minimal* element of X if:

$$m\in X \;\wedge\; (\forall x : x{\in}X : \neg(x{\sqsubset}m)) \quad .$$

□

Note that by definition $\neg(m{\sqsubset}x)$ is equivalent to $m{\sqsubseteq}x \Rightarrow x=m$, so maximality of m can be reformulated as $m\in X \;\wedge\; (\forall x : x{\in}X : m{\sqsubseteq}x \Rightarrow x=m)$, and similar for minimality.

Notice the difference between the notions "maximum" and "maximal". A value $m{\in}P$ is X's maximum if $m{\in}X$ and all elements in X are under m, whereas m is a maximal element of X if $m{\in}X$ and X contains no elements strictly above m.

A subset of a partially ordered set does not necessarily contain such extreme elements. The next lemma states that if it exists, however, the maximum of a subset is unique, and so is the minimum, if it exists. If subset X has a maximum we denote it by $\mathsf{max}X$, and its minimum we denote by $\mathsf{min}X$.

4.8 **Lemma.** Let X be a subset of a poset $(P,\sqsubseteq)$. If m and n both are X's maximum then $m=n$. If m and n both are X's minimum then $m=n$.
□

Proof. Assume that both m and n are a maximum of X, then by definition $m, n \in X$ and $(\forall x : x \in X : x \sqsubseteq m \wedge x \sqsubseteq n)$. So we both have $n \sqsubseteq m$ and $m \sqsubseteq n$. Since $\sqsubseteq$ is anti-symmetric we conclude $m = n$. $\square$

A subset of a partially ordered set does not necessarily contain maximal or minimal elements. Such elements are not unique either: a subset may have many maximal or minimal elements. As a rather trivial example, consider poset (P, I_P), where the identity relation I_P is the smallest possible partial order on P: $x\, I_P\, y \Leftrightarrow x = y$. With this particular order all elements of a subset $X \subseteq P$ both are maximal and minimal.

4.9 Definition. Let $(P, \sqsubseteq)$ be a poset. If the whole set P has a minimum this is often denoted by $\perp$ ("bottom"), and if P has a maximum this is often denoted by $\top$ ("top"). If P has a minimum the minimal elements of $P \backslash \{\perp\}$ are called P's *atoms*.

4.10 Example.

- If we consider the poset of all subsets of a set V, then the empty set $\emptyset$ is the minimum of the poset, whereas the whole set V is the maximum. The atoms are the subsets of V containing just a single element.
- In the poset $(\mathbb{N}^+, \mid)$ the whole set, $\mathbb{N}^+$, has no maximum. Its minimum, however, equals 1. The atoms are the prime numbers. Surprisingly, in the poset $(\mathbb{N}, \mid)$ the whole set $\mathbb{N}$ has a maximum: the element 0, since by definition every number is a divisor of 0.
- If $(P, \sqsubseteq)$ is totally ordered then subset $\{x, y\}$ has a maximum and a minimum, for all $x, y \in P$.

$\square$

4.11 Lemma. In a poset $(P, \sqsubseteq)$ for every subset X its maximum, if it exists, is a maximal element of X; also, its minimum, if it exists, is a minimal element of X.

Proof. Assume m is the maximum of X. Then $m \in X$ and $x \sqsubseteq m$ for all $x \in X$. Choose $x \in X$ satisfying $m \sqsubseteq x$. Then using antisymmetry and $x \sqsubseteq m$ we conclude $x = m$, so proving $m \sqsubseteq x \Rightarrow x = m$. This proves that m is maximal.

The proof for minimal/minimum is similar. $\square$

4.12 Lemma. If a poset $(P, \sqsubseteq)$ is a total order then every subset $X \subseteq P$ has at most one maximal element, which then also is its maximum. Also, X has at most one minimal element, which then also is its minimum.

Proof. Assume that m is maximal in X and $n \in X$. Then by definition $\neg(m \sqsubset n)$, so $m = n$ or $\neg(m \sqsubseteq n)$. If $\neg(m \sqsubseteq n)$ then from totality we conclude $m \sqsubseteq n$. So in both cases we have $m \sqsubseteq n$. This proves that m is the maximum of X. If moreover n is maximal too, we similarly prove $n \sqsubseteq m$, so by antisymmetry we conclude $m = n$.

The proof for minimal/minimum is similar. $\square$

4.13 Lemma. Let $(P, \sqsubseteq)$ be a *nonempty* and *finite* poset. Then P contains a maximal element and a minimal element.

Proof. Choose $x_1 \in P$ arbitrary. For $i = 1, 2, 3, \ldots$ choose x_{i+1} such that $x_i \sqsubset x_{i+1}$, that is, $x_i \sqsubseteq x_{i+1}$ and $x_i \neq x_{i+1}$. If at some point no such x_{i+1} exists, we have found a maximal element x_i.

Otherwise, the process goes on forever yielding an infinite sequence

$$x_1 \sqsubset x_2 \sqsubset x_3 \sqsubset x_4 \sqsubset x_5 \sqsubset \cdots .$$

Since P is finite, not all x_i can be distinct, so there is some $i < j$ such that $x_i = x_j$. So $x_i \sqsubseteq^* x_{j-i} \sqsubseteq x_j = x_i$. For a transitive relation R one easily proves by induction $R^n \subseteq R$ for $n > 0$. So if $i < j - 1$ then $x_i \sqsubseteq x_{j-1}$. If $i = j - 1$ then $x_i = x_{j-i}$, so in all cases we conclude $x_i \sqsubseteq x_{j-1}$. So $x_i \sqsubseteq x_{j-1} \sqsubseteq x_j = x_i$. Now antisymmetry yields $x_{j-1} = x_j$, contradicting the requirement $x_i \neq x_{i+1}$ in the construction for all i.

□

4.14 Theorem. [**Topological sorting**] For any finite poset $(P, \sqsubseteq)$ with $\#P = n$ there exists $f : P \to \{1, 2, \ldots, n\}$ such that $x \sqsubset y \Rightarrow f(x) < f(y)$ for all $x, y \in P$.

Proof. We prove the theorem by induction on $n = \#P$. For $n = 1$ we define $f(x) = 1$ for the single element x of P.

For $n > 1$ we choose $p \in P$ such that p is maximal in P; this is possible by Lemma 4.13. By the induction hypothesis there exists $f' : P \setminus \{p\} \to \{1, \ldots, n-1\}$ such that $f'(x) < f'(y)$ for all $x, y \in P \setminus \{p\}$. Next we define $f : P \to \{1, \ldots, n\}$ by $f(p) = n$ and $f(x) = f'(x)$ for $x \neq p$. It remains to prove that $f(x) < f(y)$ for all $x, y \in P$ satisfying $x \sqsubset y$, which we do by case analysis:

- $x = p$ does not occur since $x \sqsubset y$ contradicts maximality of $x = p$,
- if $y = p$ then $x \in P \setminus \{p\}$, so

$$f(x) = f'(x) \leq n - 1 < n = f(p) = f(y),$$

- if $x \neq p \neq y$ then $f(x) = f'(x) < f'(y) = f(y)$.

□

Topological sorting justifies the existence of Hasse diagrams: if every node x is drawn on height $f(x)$, then for every $x \sqsubset y$ the element y is drawn higher than x.

4.15 Example. Topological sorting has various applications. For example consider a (so-called) *spreadsheet*. In a spreadsheet the values in various *cells* depend on each other, but, in a correct spreadsheet, in an acyclic way only. The value in any particular cell in the spreadsheet can only be computed if the values in all cells on which this particular cell depends have been computed already. Therefore, an efficient implementation of these computations requires that they are performed in the "right" order. This gives rise to a partial order on the set of cells within a spreadsheet. By topological sorting the set of cells can be linearized in such a way that every cell precedes all cells depending on it; thus the computations of the values in the cells can be performed in a linear order.

4.3 Upper and lower bounds

4.16 Definition. For any subset X of a poset $(P, \sqsubseteq)$ we define, for all $m \in P$:

(a) m is an *upper bound* of X if: $(\forall x : x \in X : x \sqsubseteq m)$

(b) m is a *lower bound* of X if: $(\forall x : x \in X : m \sqsubseteq x)$

□

4.17 Properties.

(a) If P has a maximum $\top$ then $\top$ is an upper bound of every subset of P.

(b) If P has a minimum $\perp$ then $\perp$ is a lower bound of every subset of P.

(c) Every element in P is an upper bound and a lower bound of $\emptyset$.

(d) If it exists $\mathsf{max}\, X$ is an upper bound of X, for all $X \subseteq P$.

(e) If it exists $\mathsf{min}\, X$ is a lower bound of X, for all $X \subseteq P$.

□

For any subset $X \subseteq P$ we can consider the set $\{\, m \in P \mid (\forall x : x \in X : x \sqsubseteq m)\,\}$ of *all* upper bounds of X. This set may or may not have a minimum. If it has a minimum, this minimum is called the *supremum* of X, notation $\mathsf{sup}\, X$. Alternatively, it is sometimes also called X's *least upper bound*, notation $\mathsf{lub}\, X$.

Similarly, we can consider the set $\{\, m \in P \mid (\forall x : x \in X : m \sqsubseteq x)\,\}$ of *all* lower bounds of X. This set may or may not have a maximum. If it has a maximum, this maximum is called the *infimum* of X, notation $\mathsf{inf}\, X$. Alternatively, it is sometimes also called X's *greatest lower bound*, notation $\mathsf{glb}\, X$.

By combination of the definitions of maximum/minimum and of upper/lower bounds we can define supremum and infimum in a more direct way.

4.18 Definition. For any subset X of a poset $(P, \sqsubseteq)$ we define, for all $m \in P$:

(a) m is X's *supremum* if both:

$(\forall x : x \in X : x \sqsubseteq m)$, and:

$(\forall x : x \in X : x \sqsubseteq z) \;\Rightarrow\; m \sqsubseteq z$, for all $z \in P$.

Notice that the first requirement expresses that m is an upper bound of X, and that the second one expresses that m is under all upper bounds of X.

(b) m is X's *infimum* if both:

$(\forall x : x \in X : m \sqsubseteq x)$, and:

$(\forall x : x \in X : z \sqsubseteq x) \;\Rightarrow\; z \sqsubseteq m$, for all $z \in P$.

□

4.19 Example.

- For a set V its power set $\mathcal{P}(V)$ – the set of all subsets of V – with relation $\subseteq$ is a poset, and any subset X of $\mathcal{P}(V)$ has a supremum, namely $(\bigcup_{U : U \in X} U)$,and an infimum, namely $(\bigcap_{U : U \in X} U)$.
- The set $\mathbb{N}^+$ of positive natural numbers with relation $|$ ("divides") is a poset. The supremum of two elements $a, b \in \mathbb{N}^+$ is their *least common multiple*, that is, the *smallest* of all positive naturals m satisfying $a|m$ and $b|m$; usually, this value is denoted by $lcm(a, b)$.

  Similarly, the greatest common divisor of a and b , denoted by $\gcd(a, b)$, is the infimum of $\{a , b\}$.

□

4.20 Lemma. For poset $(P, \sqsubseteq)$ and for $p \in P$ we have: $\mathsf{sup}\{p\} = p$ and $\mathsf{inf}\{p\} = p$.

Proof. By Definition 4.18, to prove $\mathsf{sup}\{p\} = p$ we must prove:

$$
\begin{array}{ll}
 & (\forall x : x \in \{p\} : x \sqsubseteq p) \\
\Leftrightarrow & \quad \{ \text{ definition of } \{p\} \text{ } \} \\
 & (\forall x : x = p : x \sqsubseteq p) \\
\Leftrightarrow & \quad \{ \text{ 1-pt. rule } \} \\
 & p \sqsubseteq p \\
\Leftrightarrow & \quad \{ \sqsubseteq \text{ is reflexive } \} \\
 & \mathsf{true} \;\; ,
\end{array}
$$

and, for all $z \in P$:

$$
\begin{array}{ll}
 & (\forall x : x \in \{p\} : x \sqsubseteq z) \\
\Leftrightarrow & \quad \{ \text{ same steps as above } \} \\
 & p \sqsubseteq z \;\; ,
\end{array}
$$

which is the desired result.

□

4.21 Lemma. In a poset $(P, \sqsubseteq)$ any subset $X \subseteq P$ for which $\mathsf{sup}\, X$ exists satisfies, for all $m \in P$:

$$m = \mathsf{max}\,X \;\Leftrightarrow\; m = \mathsf{sup}\,X \;\wedge\; m{\in}X \quad ,$$

similarly, if $\mathsf{inf}\,X$ exists then, for all $m{\in}P$:

$$m = \mathsf{min}\,X \;\Leftrightarrow\; m = \mathsf{inf}\,X \;\wedge\; m{\in}X \quad .$$

Proof. By direct application of the definitions of max and sup, and of min and inf respectively.
□

Corollary: If poset $(P, \sqsubseteq)$ is such that $\mathsf{sup}\,P$ exists then $\mathsf{sup}\,P = \mathsf{max}\,P$, and if $\mathsf{inf}\,P$ exists then $\mathsf{inf}\,P = \mathsf{min}\,P$.
□

An important difference between supremum and maximum of a set is that a set's supremum may or may not be an element of that set, whereas a set's maximum always is an element of that set. The above lemma states, however, that if a set's supremum is in that set, then this supremum also is the set's maximum.

The following lemma provides a different but equivalent characterization of supremum and infimum that occasionally turns out to be very useful.

4.22 Lemma. Let X be a subset of a poset $(P, \sqsubseteq)$. For $m{\in}P$ we have:

(a) m is X's *supremum* if and only if:

$$(\forall z : z{\in}P : \, m \sqsubseteq z \Leftrightarrow (\forall x : x{\in}X : x{\sqsubseteq}z)\,) \quad .$$

(b) m is X's *infimum* if and only if:

$$(\forall z : z{\in}P : \, z \sqsubseteq m \Leftrightarrow (\forall x : x{\in}X : z{\sqsubseteq}x)\,) \quad .$$

Proof. We prove (a) only; the proof for (b) follows, mutatis mutandis, the same pattern. Firstly, we let m be X's supremum according to Definition 4.18. Then, for $z{\in}P$ we prove the equivalence of $m \sqsubseteq z$ and $(\forall x : x{\in}X : x{\sqsubseteq}z)$, by "cyclic implication":

$$m \sqsubseteq z$$

$\Leftrightarrow$ { Definition 4.18: m is an upper bound of X }

$$(\forall x : x{\in}X : x{\sqsubseteq}m) \;\wedge\; m \sqsubseteq z$$

$\Rightarrow$ { $\forall$ introduction }

$$(\forall x : x{\in}X : x{\sqsubseteq}m) \;\wedge\; (\forall x : x{\in}X : m{\sqsubseteq}z)$$

$\Leftrightarrow$ { combining terms }

$$(\forall x : x{\in}X : x{\sqsubseteq}m \wedge m{\sqsubseteq}z)$$

$\Rightarrow$ { $\sqsubseteq$ is transitive }

$$
\begin{array}{ll}
 & (\forall x : x \in X : x \sqsubseteq z\,) \\
\Rightarrow & \quad \{ \text{ Definition 4.18: } m \text{ is under all upper bounds } \} \\
 & m \sqsubseteq z \quad .
\end{array}
$$

Secondly, let m satisfy:

(2) $\quad (\forall z : z \in P : \; m \sqsubseteq z \Leftrightarrow (\forall x : x \in X : x \sqsubseteq z\,)\,) \quad .$

Then we must prove that m is X's supremum. Well, m is an upper bound:

$$
\begin{array}{ll}
 & (\forall x : x \in X : x \sqsubseteq m\,) \\
\Leftrightarrow & \quad \{ \; (2) \text{ , with } z := m \; \} \\
 & m \sqsubseteq m \\
\Leftrightarrow & \quad \{ \; \sqsubseteq \text{ is reflexive } \} \\
 & \mathsf{true} \quad ,
\end{array}
$$

and that m is under all upper bounds follows directly from (2), because $\Leftrightarrow$ is stronger than $\Rightarrow$.
□

4.23 Properties.

(a) If P has a maximum $\top$ then $\top = \mathsf{sup}\,P$ and $\top = \mathsf{inf}\,\varnothing$.

(b) If P has a minimum $\bot$ then $\bot = \mathsf{inf}\,P$ and $\bot = \mathsf{sup}\,\varnothing$.

□

4.4 Lattices

4.4.1 Definition

In the previous section we have introduced the notions of *supremum* –least upper bound– and *infimum* –greatest lower bound– of subsets of a poset. Generally, such a subset does not have a supremum or an infimum, just as, generally, not every subset has a maximum or a minimum. (Recall Lemma 4.21, for the relation between supremum and maximum, and between infimum and minimum, respectively.)

Partially ordered sets in which particular subsets do have suprema and/or infima are of interest. We will study three types of such posets, called “lattices”, “complete lattices”, and “complete partial orders”.

4.24 Definition. A poset $(P, \sqsubseteq)$ is a *lattice*, if for all $x, y \in P$ the subset $\{x\,,y\}$ has a supremum and an infimum. Because this pertains to two-element sets, it is customary to use infix-notation to denote their suprema and infima. For this purposes binary operators $\sqcup$ (“cup”) and $\sqcap$ (“cap”) are used: the supremum of $\{x\,,y\}$ then is written as $x \sqcup y$ and its infimum as $x \sqcap y$.
□

4.25 Example. Here are some examples of lattices we already encountered before.

- $(\mathbb{R}, \leq)$ is a lattice. For $x, y \in \mathbb{R}$ we have: $x \sqcup y = x \,\mathsf{max}\, y$ and $x \sqcap y = x \,\mathsf{min}\, y$.
- For a set V the poset $(\mathcal{P}(V), \subseteq)$ is a lattice, with $\sqcup = \cup$ and $\sqcap = \cap$.
- The poset $(\mathbb{N}^+, |)$ is a lattice, with $a \sqcup b = lcm(a,b)$ and $a \sqcap b = \gcd(a,b)$.

□

The following lemma actually is a special case of Lemma 4.22, namely for non-empty subsets with *at most two* elements; that is, this is Lemma 4.22 with $X := \{x, y\}$.

4.26 Lemma. In every lattice $(P, \sqsubseteq)$ we have, for all $x, y, z \in P$:

(a) $x \sqcup y \sqsubseteq z \;\Leftrightarrow\; x \sqsubseteq z \,\wedge\, y \sqsubseteq z$;

(b) $z \sqsubseteq x \sqcap y \;\Leftrightarrow\; z \sqsubseteq x \,\wedge\, z \sqsubseteq y$.

□

Actually, this lemma can serve as an *alternative definition* of $\sqcup$ and $\sqcap$, as the original definition follows from it. As a special case, we obtain the following lemma, expressing that $x \sqcup y$ is an upper bound and that $x \sqcap y$ is a lower bound.

4.27 Lemma. In every lattice $(P, \sqsubseteq)$ we have, for all $x, y \in P$:

(a) $x \sqsubseteq x \sqcup y \,\wedge\, y \sqsubseteq x \sqcup y$;

(b) $x \sqcap y \sqsubseteq x \,\wedge\, x \sqcap y \sqsubseteq y$.

Proof. Instantiate Lemma 4.26 with $z := x \sqcup y$ and $z := x \sqcap y$, respectively.
□

4.28 Lemma. In every lattice $(P, \sqsubseteq)$ we have, for all $x, y \in P$:

(a) $x \sqsubseteq y \;\Leftrightarrow\; x \sqcup y = y$;

(b) $x \sqsubseteq y \;\Leftrightarrow\; x \sqcap y = x$.

Proof. We prove (a) only, by calculation:

$$
\begin{array}{ll}
 & x \sqcup y = y \\
\Leftrightarrow & \quad \{ \sqsubseteq \text{ is reflexive and antisymmetric } \} \\
 & x \sqcup y \sqsubseteq y \,\wedge\, y \sqsubseteq x \sqcup y \\
\Leftrightarrow & \quad \{ x \sqcup y \text{ is an upper bound of } y \} \\
 & x \sqcup y \sqsubseteq y \\
\Leftrightarrow & \quad \{ \text{Lemma 4.26, with } z := y \} \\
 & x \sqsubseteq y \,\wedge\, y \sqsubseteq y \\
\Leftrightarrow & \quad \{ \sqsubseteq \text{ is reflexive } \} \\
 & x \sqsubseteq y
\end{array}
$$

□

4.4.2 Algebraic properties

The lattice operators have interesting algebraic properties, as reflected by the following theorem.

4.29 Theorem. Let $(P, \sqsubseteq)$ be a lattice. Then for all $x, y, z \in P$ we have:

(a) $x \sqcup x = x$ and $x \sqcap x = x$: $\sqcup$ and $\sqcap$ are *idempotent*;

(b) $x \sqcup y = y \sqcup x$ and $x \sqcap y = y \sqcap x$: $\sqcup$ and $\sqcap$ are *commutative*;

(c) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ and $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$: $\sqcup$ and $\sqcap$ are *associative*;

(d) $x \sqcup (x \sqcap y) = x$ and $x \sqcap (x \sqcup y) = x$: *absorption*.

Proof.

(a) See Lemma 4.20.

(b) By symmetry: set $\{x, y\}$ equals set $\{y, x\}$.

(c) By (Lemma 4.6), for all $w \in P$:

$$
\begin{array}{ll}
 & x \sqcup (y \sqcup z) \sqsubseteq w \\
\Leftrightarrow & \qquad \{ \text{Lemma 4.26} \} \\
 & x \sqsubseteq w \ \wedge \ y \sqcup z \sqsubseteq w \\
\Leftrightarrow & \qquad \{ \text{Lemma 4.26} \} \\
 & x \sqsubseteq w \ \wedge \ y \sqsubseteq w \ \wedge \ z \sqsubseteq w \\
\Leftrightarrow & \qquad \{ \text{Lemma 4.26} \} \\
 & x \sqcup y \sqsubseteq w \ \wedge \ z \sqsubseteq w \\
\Leftrightarrow & \qquad \{ \text{Lemma 4.26} \} \\
 & (x \sqcup y) \sqcup z \sqsubseteq w \quad .
\end{array}
$$

(d) We prove $x \sqcup (x \sqcap y) = x$ only, by calculation:

$$
\begin{array}{ll}
 & x \sqcup (x \sqcap y) = x \\
\Leftrightarrow & \qquad \{ \text{Lemma 4.28} \} \\
 & x \sqcup (x \sqcap y) \sqsubseteq x \\
\Leftrightarrow & \qquad \{ \text{Lemma 4.26} \} \\
 & x \sqsubseteq x \ \wedge \ x \sqcap y \sqsubseteq x \\
\Leftrightarrow & \qquad \{ \sqsubseteq \text{ is reflexive, and } x \sqcap y \text{ is a lower bound of } x \} \\
 & \mathsf{true} \quad .
\end{array}
$$

□

Conversely, the following theorem expresses that every structure with operators having the above algebraic properties "is" – can be extended into – a lattice.

4.30 Theorem. Let P be a set with two binary operators $\sqcup$ and $\sqcap$; that is, these operators have type $P \times P \to P$. Let these operators have algebraic properties (a) through (d), as in the previous theorem. Then the relation $\sqsubseteq$, on P and defined by $x \sqsubseteq y \;\Leftrightarrow\; x \sqcup y = y$ for all $x, y \in P$, is a partial order, and $(P, \sqsubseteq)$ is a lattice.
□

A direct consequence of Theorem 4.29, particularly of the associativity of the operators, is that every *finite* and *non-empty* subset of a lattice has a supremum and an infimum. Notice that the requirement "non-empty" is essential here: in a lattice the empty set may not have a supremum or infimum.

4.31 Theorem. Let $(P, \sqsubseteq)$ be a lattice. Then every finite and non-empty subset of P has a supremum and an infimum.

Proof. By Mathematical Induction on the size of the subsets, and using Theorem 4.29.
□

4.4.3 Distributive lattices

The prototype example of a lattice is the poset of all subsets of a set V , with $\subseteq$ as the partial order relation. As already mentioned, in this lattice set union, $\cup$, and intersection, $\cap$, are the binary lattice operators. In this particular lattice, the operators have an additional algebraic property, namely (mutual) *distributivity*; that is, "$\cup$ distributes over $\cap$" and "$\cap$ distributes over $\cup$", respectively:

$$X \cup (Y \cap Z) \;=\; (X \cup Y) \cap (X \cup Z) \quad \text{, for all } X, Y, Z \subseteq V \;\; ;$$

$$X \cap (Y \cup Z) \;=\; (X \cap Y) \cup (X \cap Z) \quad \text{, for all } X, Y, Z \subseteq V \;\; .$$

Generally, lattices do *not* have these properties. They do satisfy, however, a weaker version of these properties, namely "in one direction" only.

4.32 Theorem. Let $(P, \sqsubseteq)$ be a lattice. Then for all $x, y, z \in P$ we have:

(a) $x \sqcup (y \sqcap z) \;\sqsubseteq\; (x \sqcup y) \sqcap (x \sqcup z)$;

(b) $(x \sqcap y) \sqcup (x \sqcap z) \;\sqsubseteq\; x \sqcap (y \sqcup z)$.

Proof.

(a) By calculation:

$$
\begin{array}{ll}
 & x \sqcup (y \sqcap z) \;\sqsubseteq\; (x \sqcup y) \sqcap (x \sqcup z) \\
\Leftrightarrow & \quad \{ \text{ Lemma 4.26(b) } \} \\
 & x \sqcup (y \sqcap z) \;\sqsubseteq\; x \sqcup y \quad \wedge \quad x \sqcup (y \sqcap z) \;\sqsubseteq\; x \sqcup z \\
\Leftrightarrow & \quad \{ \text{ Lemma 4.26(a) (twice) } \} \\
 & x \sqsubseteq x \sqcup y \;\;\wedge\;\; y \sqcap z \sqsubseteq x \sqcup y \;\;\wedge\;\; x \sqsubseteq x \sqcup z \;\;\wedge\;\; y \sqcap z \sqsubseteq x \sqcup z \\
\Leftrightarrow & \quad \{ \text{ Lemma 4.27(a) (twice) } \} \\
 & y \sqcap z \sqsubseteq x \sqcup y \;\;\wedge\;\; y \sqcap z \sqsubseteq x \sqcup z \\
\Leftarrow & \quad \{ \sqsubseteq \text{ is transitive (twice) } \} \\
 & y \sqcap z \sqsubseteq y \;\;\wedge\;\; y \sqsubseteq x \sqcup y \;\;\wedge\;\; y \sqcap z \sqsubseteq z \;\;\wedge\;\; z \sqsubseteq x \sqcup z \\
\Leftrightarrow & \quad \{ \text{ Lemma 4.27(a) (twice) and Lemma 4.27(b) (twice) } \} \\
 & \mathsf{true} \;\; .
\end{array}
$$

(b) By duality.

□

As we have seen, some lattices –like $(\mathcal{P}(V), \subseteq)$– do satisfy the distribution properties. Such lattices are called "distributive".

4.33 Definition. A *distributive lattice* is a lattice $(P, \sqsubseteq)$ in which $\sqcup$ and $\sqcap$ distribute over each other; that is, for all $x, y, z \in P$:

(a) $x \sqcup (y \sqcap z) \;=\; (x \sqcup y) \sqcap (x \sqcup z)$;

(b) $x \sqcap (y \sqcup z) \;=\; (x \sqcap y) \sqcup (x \sqcap z)$.

□

4.34 Example. Not every lattice is distributive. The smallest example illustrating this has 5 elements $\top, a, b, c, \bot$, say, with this partial order: $\bot$ is under all elements, a, b, c are under $\top$ and are mutually incomparable. (See the Hasse diagram in Figure 24.) In this lattice $a \sqcup (b \sqcap c) \;=\; a$ whereas $(a \sqcup b) \sqcap (a \sqcup c) \;=\; \top$.

4.4.4 Complete lattices

4.35 Definition. A *complete lattice* is a partial order in which *every* subset has a supremum and an infimum. In particular, the whole set has a supremum and an infimum, usually denoted by $\top$ and $\bot$, respectively. Notice that $\top$ and $\bot$ also are the lattice's maximum and minimum. (Recall Property 4.23.)
□

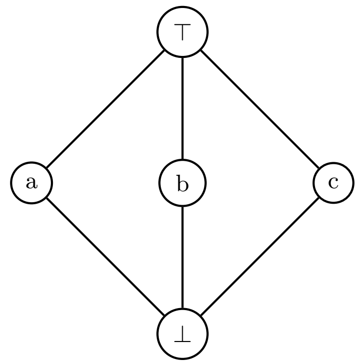


Figure 24: The smallest non-distributive lattice

4.36 Lemma. Every *finite* lattice is complete.

Proof. Let $(P, \sqsubseteq)$ be a finite lattice. By Theorem 4.31 this lattice is *almost* complete already: every non-empty subset has a supremum and an infimum, so all we must do is prove that $\emptyset$ has a supremum and an infimum. From Property 4.23 we know that $\mathsf{sup}\,\emptyset = \bot$ and $\mathsf{inf}\,\emptyset = \top$.
□

4.37 Example. The power set $(\mathcal{P}(V), \subseteq)$ of all subsets of a set V is a complete lattice. The supremum of a set X of subsets of V is the union of all sets in X, which is $(\bigcup_{U:U\in X} U)$; its infimum is the intersection of all its elements, which is $(\bigcap_{U:U\in X} U)$.
□

4.38 Example. The poset $(\mathbb{R}, \leq)$ is a lattice, but it is not complete, but every *closed* interval $[a, b]$, with $a \leq b$ and with the same partial order $\leq$, is a complete (sub)lattice.
□

Completeness is a strong property, so strong, actually, that we only have to prove half of it: if every subset has an infimum that it also has a supremum, so if every subset has an infimum the partial order is a complete lattice.

4.39 Theorem. Let $(P, \sqsubseteq)$ be a poset. Then:

(a) "Every subset of P has an infimum" $\Rightarrow$ "$(P, \sqsubseteq)$ is a complete lattice" ;

(b) "Every subset of P has a supremum" $\Rightarrow$ "$(P, \sqsubseteq)$ is a complete lattice" .

Proof. We prove (a) only. To prove this we assume that every subset of P has an infimum. Then, to prove that $(P, \sqsubseteq)$ is complete we only must prove that every subset of P has a supremum. So, let X be a subset of P. We define a subset Y by $Y = \{\, y \in P \mid (\forall x : x \in X : x \sqsubseteq y)\,\}$, that is, Y is the set of all upper bounds of X. Now let $m = \mathsf{inf}\, Y$. We prove that this m is X's supremum.

"m is an upper bound of X":

$$
\begin{array}{ll}
 & (\forall x : x \in X : x \sqsubseteq m) \\
\Leftarrow & \qquad \{\ m \text{ is } Y\text{'s } \textit{greatest} \text{ lower bound }\} \\
 & (\forall x : x \in X : (\forall y : y \in Y : x \sqsubseteq y)\,) \\
\Leftrightarrow & \qquad \{\ \text{exchanging dummies }\} \\
 & (\forall y : y \in Y : (\forall x : x \in X : x \sqsubseteq y)\,) \\
\Leftrightarrow & \qquad \{\ \text{definition of } Y\ \} \\
 & (\forall y : y \in Y : y \in Y) \\
\Leftrightarrow & \qquad \{\ \text{predicate calculus }\} \\
 & \mathsf{true}\ .
\end{array}
$$

"m is under all upper bounds of X": For any $y \in P$ we derive:

$$
\begin{array}{ll}
 & (\forall x : x \in X : x \sqsubseteq y) \\
\Leftrightarrow & \qquad \{\ \text{definition of } Y\ \} \\
 & y \in Y \\
\Rightarrow & \qquad \{\ m \text{ is a lower bound of } Y\ \} \\
 & m \sqsubseteq y\ .
\end{array}
$$

□

remark: In the proof of this theorem we introduced set Y as the set of *all* upper bounds of X. It is possible that X has no upper bounds, in which case Y is *empty*. This is harmless, because if *every* subset has an infimum then so has the empty set. Thus, this proof crucially depends on the property that also the empty set has an infimum. Recall that in a non-complete lattice the empty set does not need to have an infimum or supremum. (Also see the discussion preceding Theorem 4.31.)

□

4.5 Exercises

1. Consider the poset $(\mathbb{N}^+,\ |\ )$. Let $A = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

   (a) Establish all minimal and maximal elements of A.

   (b) Give a subset of four elements of A that has a maximum.

   (c) Give a subset of four elements of A that has a minimum.

2. Draw the Hasse diagram of

$$\{X \subseteq \{1,2,3,4\} \mid 2 \in X \vee 3 \notin X\}$$

with respect to the partial order $\subseteq$.

3. Let $(P, \sqsubseteq)$ be a poset. Recall that for all $x, y \in P$:

$$x \sqsubset y \ \Leftrightarrow \ x \sqsubseteq y \ \wedge \ x \neq y \ .$$

Prove that the relation $\sqsubset$ is irreflexive, antisymmetric, and transitive.

4. Let $(P, \sqsubseteq)$ be a poset. Prove that for all $x, y \in P$:

(a) $x \sqsubseteq y \ \Leftrightarrow \ (\forall z : z \in P : x \sqsubseteq z \Leftarrow y \sqsubseteq z)$;

(b) $x \sqsubseteq y \ \Leftrightarrow \ (\forall z : z \in P : z \sqsubseteq x \Rightarrow z \sqsubseteq y)$.

5. Let $(P, \sqsubseteq)$ be a poset and X a subset of P. Prove that an element $m \in X$ is maximal if and only if for all $x \in X$ we have $m \sqsubseteq x \Rightarrow m = x$.

6. A poset $(U, \sqsubseteq)$ is given with two subsets X and Y for which $\sup(X)$ and $\sup(Y)$ exist, and $\sup(X) \in Y$. Prove that $\sup(Y)$ is an upper bound of X.

7. Let $(U, \sqsubseteq)$ be a poset and $f : U \to U$ a function. Define the relation $\leq$ on U by

$$x \leq y \iff f(x) \sqsubseteq f(y).$$

(a) Prove that $(U, \leq)$ is a poset if f is injective.

(b) Give an example of a poset $(U, \sqsubseteq)$ and a non-injective function $f : U \to U$ such that $(U, \leq)$ is not a poset.

8. Consider the poset $(\mathbb{N}^+, \mid)$.

(a) Establish the supremum and the infimum of $\{1, 2, 3, 4, 5\}$.

(b) Establish the supremum and the infimum of $\{1, 2, 3, 4, 6, 12\}$.

(c) Establish the supremum and the infimum of $\{3, 4, 5, 15, 20, 60\}$.

(d) Establish the supremum and the infimum of $\{3, 4, 5, 12\}$.

(e) Establish the supremum and the infimum of the set of all even numbers.

For all set also establish whether the set has a minimum and/or a maximum.

9. Let $(U, \sqsubseteq)$ be a poset. Two sets $A, B \subseteq U$ are given, for which $\sup(A)$ exists, and for which $b \in B$ is minimal in B and $\sup(A) \sqsubseteq b$. Prove that $A \cap B \subseteq \{b\}$.

10. Let $(P, \sqsubseteq)$ be a lattice and $x, y, z \in P$. Prove that:

(a) $x \sqsubseteq y \ \Rightarrow \ x \sqcup z \sqsubseteq y \sqcup z$;

(b) $x \sqsubseteq z \ \Rightarrow \ z \sqcup (x \sqcap y) = z$.

11. We consider the poset $(\mathbb{Q}, \leq)$.

    (a) Prove that the set $\{x \in \mathbb{Q} \mid x < 1\}$ has no maximum.

    (b) Prove that this set has a supremum.

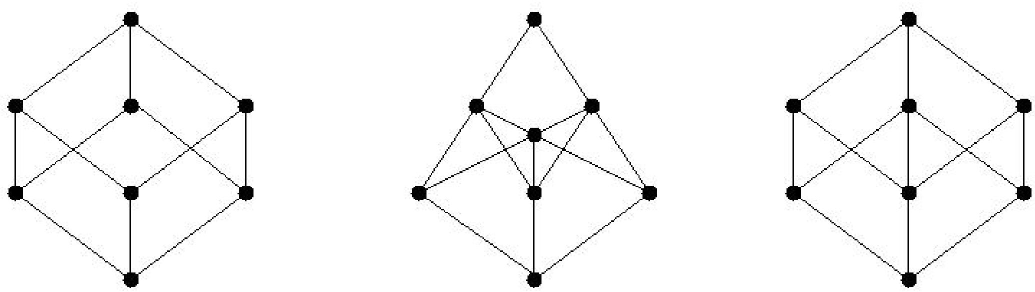
12. We consider a poset $(P, \sqsubseteq)$; let X and Y be subsets of P such that $\mathsf{sup}\, X$ and $\mathsf{inf}\, Y$ both exist. In addition it is given that $x \sqsubseteq y$, for all $x \in X$ and $y \in Y$. Prove that $\mathsf{sup}\, X \sqsubseteq \mathsf{inf}\, Y$.

13. We consider a poset $(P, \sqsubseteq)$; let X be a subset of P.

    (a) Prove that if X contains two (different) maximal elements then X has no maximum.

    (b) Prove that if $(\forall x, y : x, y \in X : x \sqsubseteq y \vee y \sqsubseteq x)$ and if X contains a maximal element then X has a maximum.

14. In the figure below you see three diagrams. Which of these diagrams are Hasse diagrams? Which of these diagrams represents a lattice?



15. Show that every lattice with *at most* 4 elements is distributive.

16. Is the poset $(\mathbb{N}^+, \mid)$ a complete lattice? How about $(\mathbb{N}, \mid)$?

17. Suppose $(P, \sqsubseteq)$ is a lattice and $a, b \in P$ with $a \sqsubseteq b$. Prove that $([a, b], \sqsubseteq)$ is a lattice too. Here $[a, b]$ denotes the interval from (and including) a upto (and including) b; how would you define this interval?

18. Let $(P, \sqsubseteq)$ be a lattice. Prove that if $\sqcup$ distributes over $\sqcap$ then $\sqcap$ also distributes over $\sqcup$.

19. Prove that, in a complete lattice $(P, \sqsubseteq)$, the extreme element $\top$ is the identity element of $\sqcap$ and a zero element of $\sqcup$. Similarly, show that $\perp$ is the identity of $\sqcup$ and a zero of $\sqcap$.